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# CERTAIN CONJECTURES RELATING UNIPOTENT ORBITS TO AUTOMORPHIC REPRESENTATIONS

BY

DAVID GINZBURG

*School of Mathematical Sciences, Sackler Faculty of Exact Sciences*
*Tel Aviv University, Ramat Aviv, Tel Aviv, 69978, Israel*
*e-mail: ginzburg@math.tau.ac.il*

ABSTRACT

In this paper I formulate certain conjectures relating the structure of unipotent orbits to automorphic representations. We consider a few examples and prove some of these conjectures.

## 1. Introduction

One of the important problems in the theory of automorphic representations is to determine the set of Fourier coefficients these representations have. Knowledge of these Fourier coefficients is most crucial in many applications. For example, when one constructs a Rankin–Selberg integral, which is an integral consisting of certain automorphic forms, the Fourier coefficients of these automorphic forms are important in order to establish whether the integral is Eulerian or not. Such questions as uniqueness of a functional which is defined by a certain Fourier coefficient, are essential in determining if the integral is Eulerian or not.

In recent years, knowledge of Fourier coefficients of automorphic representations is also used to establish liftings from one algebraic group to another. For example, the descent method (see [G-R-S1]) which is used to establish the lifting between certain automorphic representations of $GL_n(\mathbf{A})$ to certain automorphic representations of classical split groups, relies heavily on knowledge of Fourier coefficients. In that method, one considers a certain residue of an Eisenstein series and studies a certain Fourier coefficient of this representation. Establishing the cuspidality of the lift and its non-vanishing relies completely

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on knowledge of which nonzero Fourier coefficients the residue has, and which Fourier coefficient the residue does not have.

The idea of connecting unipotent orbits to representations is not new and was considered in various cases by several authors. The basic reference for definitions and properties of unipotent orbits can be found in [S] or [C-M]. We will use this reference for the basic definitions we need. The idea is to use the structure of the unipotent orbits of a complex group and to associate with each unipotent orbit a set of Fourier coefficients. The way to establish this association was done in [G-R-S2] for the symplectic groups. We recall the precise definition in Section 2. Let $G$ be a split algebraic group and let $\pi$ be an automorphic representation of $G(\mathbf{A})$. Roughly, one defines the set $\mathcal{O}_G(\pi)$ as follows. Recall that the set of unipotent orbits of $G(\mathbf{C})$ has the structure of partial ordering. A unipotent orbit $\mathcal{O}$ is in $\mathcal{O}_G(\pi)$ if the following holds. For every unipotent orbit $\mathcal{O}'$ which is greater than $\mathcal{O}$, the representation $\pi$ has no nonzero Fourier coefficient associated with $\mathcal{O}'$. Also, the representation $\pi$ has a nonzero Fourier coefficient which is associated with the unipotent orbit $\mathcal{O}$. For example, if $\pi$ is a generic representation of $G(\mathbf{A}) = GL_n(\mathbf{A})$ then $\mathcal{O}_G(\pi) = (n)$. The minimal representation $\pi$ of the group $G(\mathbf{A}) = SO_{2n}(\mathbf{A})$ satisfies $\mathcal{O}_G(\pi) = (2^2 1^{2n-4})$. For more on this set, see Section 2.

As explained above, knowledge of the structure of the set $\mathcal{O}_G(\pi)$ is very important for various applications. In [M] this set was studied for groups $G$ defined over a $p$-adic field. Some of the results there were the motivations for the global analogues. For example, in [M] it is shown that if $G$ is a classical group, $\pi$ is supercuspidal and $\mathcal{O} \in \mathcal{O}_G(\pi)$, then $\mathcal{O}$ cannot be conjugated into a Levi part of a parabolic subgroup of $G$. Another result proved in [M] concerns the set of special unipotent orbits. It is proved there that if $\pi$ is an arbitrary representation of a classical group, then the set $\mathcal{O}_G(\pi)$ consists of special unipotent orbits. In [G-R-S2] some analogous results for automorphic representations are proved. For example, it is shown that if $\pi$ is an automorphic representation defined on the adele points of a classical group, then a unipotent orbit in $\mathcal{O}_G(\pi)$ must be special. In Section 3 we quote this Theorem as Theorem 3.1 and indicate the idea behind the proof.

We now give some details of the content of the paper. In Section 2 we recall some basic definitions and properties of unipotent orbits. We explain in detail how to associate with a unipotent orbit a set of Fourier coefficients. Then we give the precise definition of the set $\mathcal{O}_G(\pi)$ and define some further operations on the set of partitions needed for other sections. Sections 3, 4 and 5 are devoted to the set $\mathcal{O}_G(\pi)$. In Section 3 we study this set for arbitrary automorphic representations $\pi$, and in Section 4 we study this set for cuspidal representations $\pi$.

Section 5 is the main section of the paper. In this section we study the set $\mathcal{O}_G(\pi)$ where $\pi$ is an Eisenstein series or any of its residues. The main part is to introduce some conjectures which relate the set $\mathcal{O}_M(\tau)$ with the set $\mathcal{O}_G(\pi)$. Here $M$ is a Levi part of a parabolic subgroup of $G$, and $\tau$ is an automorphic representation defined on the group $M(\mathbf{A})$. To get a better understanding, we first study in sub-section 5.1 these connections for the group $GL_n$. In Propositions 5.2 and 5.3 we find the set $\mathcal{O}_G(\pi)$ for some important examples of representations, like the Speh representations. Then, in Definition 5.5 we define the sets $\mathcal{O}_G^{\max}$ and $\mathcal{O}_G^{\min}$. Motivated by the examples we consider, we introduce in Conjecture 5.6 our main conjecture. In this conjecture we write down the expected maximal and minimal unipotent orbits which an Eisenstein series and its residues can obtain. Based on this conjecture we prove what we refer to as the Min-Max Principle. This is Proposition 5.8, which is based on basic properties of partitions. In sub-section 5.2 we repeat, with appropriate modifications, these conjectures and relations, this time for other classical groups. The expected picture is quite similar. In sub-section 5.3 we vaguely mention the picture in the exceptional group $G_2$.

In Section 6 we define the notion of the graph of an Eisenstein series. In Section 5 we gave a conjecture of what we denoted by $\mathcal{O}_G^{\max}$ and $\mathcal{O}_G^{\min}$. It is natural to study the other unipotent orbits which correspond to other residue representations corresponding to a specific Eisenstein series. We give some examples, but since the general situation is not clear we do not state any conjectures. Instead, we propose some further possible problems related to these questions.

In Section 7 we give some examples related to the conjectures and the problems which were studied in previous sections.

Finally, in this paper, as the title suggests, our main purpose was to introduce some conjectures relating the structure of unipotent orbits with Fourier coefficients of automorphic forms. We do state and prove some examples to motivate our conjectures. Most of the proofs are based on ideas and proofs which appear in the literature and hence we prefer either to give a sketch idea of the proof, or just to refer the reader to other similar proofs.

As mentioned above, the idea of establishing a correspondence between representations and unipotent orbits is not new. It was studied in detail in many cases, especially for finite groups of Lie type. Some other definitions that we introduce, like the Gelfand–Kirillov dimension of a representation, are also not new (see, for example, [K] p. 158 and the references given there).

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## 2. Unipotent orbits and Fourier coefficients

In this section we will explain how to associate with a unipotent orbit a set of Fourier coefficients. As mentioned in the introduction this was done for symplectic groups in [G-R-S2]. We refer the reader to [C-M] for the basic notation and properties of unipotent orbits.

The set of unipotent orbits for classical groups are parameterized by partitions. For the group $GL_n$, the unipotent orbits are parameterized by the set of all partitions of $n$. For the group $Sp_{2n}$, the unipotent orbits are parameterized by all partitions of $2n$ such that odd numbers occur with even multiplicities. For the orthogonal group $SO_n$, they are parameterized by all partitions of $n$ such that even numbers occur with even multiplicities. For the exceptional groups there is the Bala–Carter parameterization of the set of unipotent orbits. Let $G$ denote a split reductive group. A partition $\mathbf{a}$ will be said to be $G$ admissible if $\mathbf{a}$ corresponds to a unipotent orbit of $G$. When $G$ is a classical group, it is convenient to identify the set of unipotent orbits of $G$ with the set of all $G$ admissible partitions.

For any classical group $G$ as above, there is the structure of partial ordering defined on the set of unipotent orbits. It is defined as follows. Suppose that $\mathbf{a} = (r_1 r_2 \ldots r_m)$ and $\mathbf{b} = (l_1 l_2 \ldots l_m)$ are two unipotent orbits for a given group $G$, where we assume that $r_1 \geq r_2 \geq \cdots \geq r_m \geq 0$ and similarly for $\mathbf{b}$. We say that $\mathbf{a} \geq \mathbf{b}$ if $r_1 + \cdots + r_i \geq l_1 + \cdots + l_i$ for all $1 \leq i \leq m$. For the exceptional groups one can find the partial ordering in [C].

We shall now explain how to associate with each unipotent orbit a set of Fourier coefficients. This set can consist of one member or can be infinite. We shall work this out in the classical groups. In the exceptional groups this correspondence is done in a similar way. Let $\mathbf{a} = (m_1 m_2 \ldots m_p)$ be a partition which corresponds to a group $G$ of rank $n$. We shall assume that $m_1 \geq m_2 \geq \cdots \geq m_p > 0$. To each $m_i$ we associate the diagonal matrix $\mathrm{diag}(t^{(m_i-1)}, t^{(m_i-3)}, \ldots, t^{-(m_i-3)}, t^{-(m_i-1)})$. Combining all such diagonal matrices and arranging them in decreasing order of $t^{k_i}$, we obtain a one dimensional torus

$$h_{\mathbf{a}}(t) = \mathrm{diag}(t^{(m_1-1)}, \ldots, t^{-(m_1-1)}).$$

Since $\mathbf{a}$ is a partition of $n$ it follows that $h_{\mathbf{a}}(t)$ can be realized as a torus element in the group $G$. For example, suppose that $\mathbf{a} = (3^2 2)$. Then $h_{\mathbf{a}}(t) = \mathrm{diag}(t^2, t^2, t, 1, 1, t^{-1}, t^{-2}, t^{-2})$.

Let $U$ denote the standard maximal unipotent subgroup of $G$. In terms of matrices we shall realize it as upper unipotent matrices. The one dimensional torus $h_{\mathbf{a}}(t)$ acts on $U$ by conjugation. Let $\alpha$ denote a positive root and let $x_\alpha(r)$ be the one dimensional unipotent subgroup in $U$ corresponding to the root $\alpha$. We have $h_{\mathbf{a}}(t)x_\alpha(r)h_{\mathbf{a}}(t)^{-1} = x_\alpha(t^i r)$ and since $\alpha$ is positive it follows that $i \geq 0$. On the subgroups $x_\alpha(r)$ which correspond to negative roots $\alpha$, the torus $h_{\mathbf{a}}(t)$ acts with non-positive powers of $t$. This creates a partition on the set of all positive roots of $G$. Denote by $U_i$ the set of all $x_\alpha(r)$ such that $h_{\mathbf{a}}(t)x_\alpha(r)h_{\mathbf{a}}(t)^{-1} = x_\alpha(t^i r)$. Let $L$ denote the subgroup of $G$ generated by all $x_{\pm\alpha}(r)$ such that $h_{\mathbf{a}}(t)x_\alpha(r)h_{\mathbf{a}}(t)^{-1} = x_\alpha(r)$. Let $C$ denote the stabilizer in $L$ of any representative of the unipotent orbit defined by $\mathbf{a}$. It is well known (see [C-M] or [C]) that $C$ is a reductive subgroup of $L$. To define the corresponding Fourier coefficients we first define the unipotent group $V$ to be the subgroup of $U$ generated by all $x_\alpha(r) \in U_i$ such that $i \geq 2$. Let $\mathbf{A}$ be the ring of adeles of a global field $F$. Let $\psi$ denote a nontrivial additive character of $F\backslash\mathbf{A}$. Let $\psi_V$ denote any nontrivial additive character of $V/[V,V]$ with points in $F\backslash\mathbf{A}$, such that the stabilizer of $\psi_V$ inside $L(F)$ is the group $C_1(F)$, and such that over $\mathbf{C}$ the group $C_1$ is of the same type as the group $C$. The group $C_1(F)$ depends on the choice of the representative of the unipotent orbit that we choose. When there is no confusion, we shall write $C$ for the group $C_1$. We extend $\psi_V$ trivially to the group $V(F)\backslash V(\mathbf{A})$.

Let $\pi$ denote an automorphic representation of the group $G(\mathbf{A})$ and let $\varphi$ denote a vector in the space of $\pi$. We define

$$\mathcal{F}_{\mathbf{a}}(\varphi)(g) = \int_{V(F)\backslash V(\mathbf{A})} \varphi(vg)\psi_V(v)dv. \tag{1}$$

We say that the above Fourier coefficient corresponds to or is associated with the unipotent orbit $\mathbf{a}$. We will say that $\pi$ has a nonzero Fourier coefficient associated with the unipotent orbit $\mathbf{a}$ if integral (1) is nonzero for some choice of data and for some choice of representative of the orbit $\mathbf{a}$. Conversely, if integral (1) is zero for every choice of data and all representatives of $\mathbf{a}$ inside $V(F)$, we will say that $\pi$ has no nonzero Fourier coefficients corresponding to the unipotent orbit $\mathbf{a}$. Let $u_{\mathbf{a}}$ denote any representative of $\mathbf{a}$ inside the group $V(F)$. We shall denote by $\mathcal{F}_{u_{\mathbf{a}}}(\varphi)$ the Fourier coefficient of $\pi$ which corresponds to this specific representative.

Before considering a few examples, let us mention that integral (1) defines an automorphic representation on the group $C(\mathbf{A})$.

*Examples:* (1) Let $G = GL_4$ and consider the unipotent orbit $\mathbf{a} = (2^2)$. Then $h_{\mathbf{a}}(t) = \mathrm{diag}(t, t, t^{-1}, t^{-1})$. In this case the group $L = GL_2 \times GL_2$ and, as can be deduced from [C] p. 398, the group $C$ equals $GL_2$ embedded diagonally inside $L$. The Fourier coefficient (1), which corresponds to $(2^2)$, can be chosen as

$$(2) \qquad \mathcal{F}_{\mathbf{a}}(\varphi)(g) = \int_{(F\backslash\mathbf{A})^4} \varphi\left(\begin{pmatrix} 1 & & x_1 & x_2 \\ & 1 & x_3 & x_4 \\ & & 1 & \\ & & & 1 \end{pmatrix} g\right) \psi(x_1 + x_4) dx_i.$$

Thus, in this case there corresponds to $\mathbf{a}$ one Fourier coefficient. All others are conjugate to this one under the action of $L(F)$.

(2) Let $G = Sp_4$ and suppose that $\mathbf{a} = (2^2)$. The torus $h_{\mathbf{a}}(t)$ is as in example (1), but this time viewed as a torus element in $G$. The group $L = GL_2$, and from [C] we deduce that $C$ is a one dimensional torus. The Fourier coefficient defined in (1) is now chosen as

$$(3) \qquad \mathcal{F}_{\mathbf{a}}(\varphi)(g) = \int_{(F\backslash\mathbf{A})^3} \varphi\left(\begin{pmatrix} 1 & & x & y \\ & 1 & z & x \\ & & 1 & \\ & & & 1 \end{pmatrix} g\right) \psi(\beta y + \gamma z) dx dy dz,$$

where $\beta, \gamma \in F^*$. This time there are an infinite number of non-conjugate Fourier coefficients associated with $(2^2)$ depending on which square classes the elements $\beta, \gamma$ are in. For example, if $\beta\gamma = -\epsilon^2$, then $C(F)$ is isomorphic to $GL_1(F)$. In other cases, the group $C(F)$ will be isomorphic to an anisotropic group $O_2(F)$ which depends on $\beta$ and $\gamma$.

Let $G$ denote a split reductive group. We recall the definition of the set $\mathcal{O}_G(\pi)$.

*Definition 2.1:* Let $\pi$ be an automorphic representation defined on the group $G(\mathbf{A})$. We define the set $\mathcal{O}_G(\pi)$ as follows. A unipotent orbit $\mathcal{O}$ is in the set $\mathcal{O}_G(\pi)$ provided the representation $\pi$ has no nonzero Fourier coefficients $\mathcal{F}_{\mathbf{a}}(\varphi)$ for all $\mathbf{a}$ which is greater than $\mathcal{O}$. Also, the representation $\pi$ has a nonzero Fourier coefficient which is associated with the unipotent orbit $\mathcal{O}$.

The notion of $\mathcal{O}_G(\pi)$ is well defined. Indeed, if $\pi$ is not the trivial representation, then $\pi$ has a nonzero Fourier coefficient associated with the minimal unipotent orbit. (This is the one orbit above the trivial orbit.) This follows from the fact that the Fourier coefficients which are associated with the minimal orbit are the Fourier coefficients along the one dimensional unipotent subgroup corresponding to the highest root of $G$.

Similarly, we define the set $\mathcal{O}_G(\pi)$ for representations on the covering groups.

We will also need the notion of a **collapse** of a unipotent orbit. Let $G$ be a classical group of rank $n$ and let $\mathbf{a}$ be any partition of $n$. Following [C-M] p. 99, we define the $G$ collapse of $\mathbf{a}$ to be the largest unipotent orbit of the group $G$ which is smaller than or equal to $\mathbf{a}$. We shall denote this unipotent orbit by $\mathbf{a}_G$. The fact that this notion is well defined follows from [C] Lemma 6.3.3. The precise definition of $\mathbf{a}_G$ is as follows. Assume that $G$ is a reductive group of type $B$. The other cases are defined similarly. Write $\mathbf{a} = (p_1 p_2 \ldots p_r)$, where $p_i \geq p_{i+1}$ and if necessary we allow zeros. If every even number in $\mathbf{a}$ occurs with even multiplicity, then $\mathbf{a}_G = \mathbf{a}$. Otherwise, let $p_i$ denote the largest even number which occurs with odd multiplicity. Let $p_j$ be the largest integer occurring in $\mathbf{a}$, such that $p_j < p_i - 1$. Notice that $p_j$ can be zero. Replace $p_i$ by $p_i - 1$ and $p_j$ by $p_j + 1$. Continue this process until we get a partition where each even number occurs with even multiplicity.

*Examples:* (1) Suppose $G$ is of type $D$ and let $\mathbf{a} = (8)$. Then $\mathbf{a}_G = (71)$. This is an example where $p_j = 0$ for some $j$.

(2) Suppose that $G$ is of type $C$ and let $\mathbf{a} = (7^3 3^3)$. We have $p_1 = 7$ and $p_j = 3$. We get $\mathbf{a}_G = (7^2 643^2)$.

We define two operations on the set of all partitions. Let $\mathbf{a} = (p_1 p_2 \ldots p_r)$ be a partition of $n$ and let $\mathbf{b} = (q_1 q_2 \ldots q_r)$ be a partition of $m$. We assume that $p_i \geq p_{i+1}$ and $q_i \geq q_{i+1}$, and by inserting zeros we may assume the same number $r$. We now define

*Definition 2.2:* With the above notation we define the addition of $\mathbf{a}$ and $\mathbf{b}$ to be the partition of $n+m$ defined as $\mathbf{a} + b = ((p_1+q_1)(p_2+q_2)\ldots(p_r+q_r))$. We also define the product of $\mathbf{a}$ and $\mathbf{b}$ to be the partition of $n+m$ defined as $\mathbf{ab} = (p_1 p_2 \ldots p_r q_1 q_2 \ldots q_r)$ and then rearranging the numbers in a decreasing order.

Let $\mathbf{a}$ denote a $G$ admissible unipotent orbit, where $G$ is a reductive group. We shall denote by $\mathbf{a}^{S(G)}$ the smallest *special* $G$ admissible orbit which is larger than $\mathbf{a}$. The fact that this orbit is unique is proved in [C-M] Lemmas 6.3.8 and 6.3.9. In these lemmas they also explain how to construct this orbit. When the group $G$ is clear, we shall denote $\mathbf{a}^{S(G)}$ by $\mathbf{a}^S$.

Finally, we recall the notion of the transpose of a partition (see [C-M] p. 65). Let $\mathbf{a} = (k_1k_2\dots k_r)$ denote a partition of $n$ where $k_i \geq k_{i+1} > 0$ for all $i$. We define the transpose of $\mathbf{a}$ to be the partition of $n$ denoted by $\mathbf{a}^t = (m_1m_2\dots m_r)$ where $m_j = |\{i : k_i \geq j\}|$. Another way to describe the transpose of a partition is using the Young diagram. In this description we associate with $\mathbf{a}$ an array which consists of $k_1$ boxes in the first row, then $k_2$ boxes in the second row, and so on. Then, the number $m_1$ is the number of boxes in the first column of this array, the number $m_2$ is the number of boxes in the second column, and so on. For example, if $\mathbf{a} = (21^3)$ then $\mathbf{a}^t = (41)$.

## 3. On the set $\mathcal{O}_G(\pi)$ for automorphic representations

Let $\pi$ be an automorphic representation defined on the group $G(\mathbf{A})$. The following result, proved in [G-R-S2] for symplectic groups, was motivated by the result of [M] for local $p$-adic fields. We refer the reader to [C-M] page 100 for the definition of special unipotent orbit.

THEOREM 3.1: *Let $G$ be a classical group. The set $\mathcal{O}_G(\pi)$ consists of unipotent orbits which are all special.*

As mentioned above, the proof in [G-R-S2] is only for the symplectic group but it is similar for the orthogonal groups. The proof in [M] is for any classical group. We briefly explain the idea of the proof.

Let $\mathbf{a}$ be a unipotent orbit in $\mathcal{O}_G(\pi)$ and assume that $\mathbf{a}$ is not special. We shall derive a contradiction. In section 2 we defined the sets $U_i$. For some unipotent orbits the set $U_1$ is not empty. If this is the case, then the group $U_{i\geq 1}/[U_{i\geq 2}, U_{i\geq 2}]$ is a generalized Heisenberg unipotent group. Here $U_{i\geq 1} = U_1U_2\dots U_r$, where $r$ is the largest integer such that $U_r$ is not zero and $U_i$ is zero for all $i > r$. Similarly, we define $U_{i\geq 2}$. The group $U_{i\geq 2}$ was denoted by $V$ in section 2. This means that the group $C$ as defined in section two has an embedding inside a suitable symplectic group. It also means that we can define a projection $\sigma$ from $U_{i\geq 1}$ onto a suitable Heisenberg group. As in [G-R-S2] p. 4 we consider the following integral

$$(4) \qquad f(h) = \int_{U_{i\geq 1}(F)\backslash U_{i\geq 1}(\mathbf{A})} \theta_\phi^\psi(\sigma(v)h)\varphi(vh)\psi_V(ker\sigma(v))dv.$$

Here $\theta_\phi^\psi$ is the theta representation defined on the double cover of a suitable symplectic group. Thus, the space of functions defined by (4) defines an automorphic representation on the group $\widetilde{C}(\mathbf{A})$, that is, on the double cover of the group $C$. This representation can be genuine or not, depending on whether the group $C$ splits under the double cover or not. It follows from the definition of special unipotent orbit that the group $C$ contains a unipotent subgroup. Also, and that's the key point, it follows that the representation defined by (4) defines a genuine representation of $\widetilde{C}(\mathbf{A})$ if and only if the unipotent orbit $\mathbf{a}$ is not special. This is also proved in [N]. Using these two facts, we consider the one dimensional unipotent subgroup $x_\alpha(r)$ which corresponds to the highest root $\alpha$ in $C$. It follows that the group $SL_2$ generated by $x_{\pm\alpha}(r)$ does not split under the double cover, and hence when we restrict (4) to this copy of $SL_2$, we obtain a genuine representation defined on the group $\widetilde{SL}_2(\mathbf{A})$. We expand (4) along the unipotent group $x_\alpha(r)$ with points in $F\backslash\mathbf{A}$. Since a genuine function on $\widetilde{SL}_2(\mathbf{A})$ cannot equal its constant term, it follows that it has at least one nontrivial Fourier coefficient. It is not hard to show that this Fourier coefficient corresponds to a unipotent orbit which is higher than $\mathbf{a}$. This gives us a contradiction to the assumption that $\mathbf{a} \in \mathcal{O}_G(\pi)$.

It should be mentioned that Theorem 3.1 is not true for covering groups. In [B-F-G] a small representation was constructed on the double cover of the odd orthogonal group. For that representation we have $\mathcal{O}_G(\pi) = (2^n1)$ or $\mathcal{O}_G(\pi) = (2^n1^3)$. These two unipotent orbits are not special.

The situation in the exceptional groups is different. There are examples of unipotent orbits which are not special but that the group $C$ will split under the double cover. This already occurs in the exceptional group $G_2$. The unipotent orbit labelled $A_1$ is not special, and the group $C$ is $SL_2$. However, the symplectic embedding of $C$ is into $Sp_4$ via the symmetric cube representation. It is well known that this embedding splits under the double cover. Nevertheless, we believe that Theorem 3.1 still holds for the exceptional group $G = G_2$. As indicated in [Sa], over local fields, the minimal representation is unique. We expect that this will also be true for automorphic representations as well. It is not clear what happens in the other exceptional groups.

In general, it is also not clear if $\mathcal{O}_G(\pi)$ can consist of more than one unipotent orbit. Clearly, if this is the case, then any two such unipotent orbits will have to be not related under the definition of the partial order.

## 4. On the set $\mathcal{O}_G(\pi)$ for cuspidal representations

In this section we will assume that $\pi$ is a cuspidal representation defined on the group $G(\mathbf{A})$. We will say that $\mathbf{a}$ is a cuspidal unipotent orbit for $\pi$ if $\mathbf{a} \in \mathcal{O}_G(\pi)$. In [M-W] the following local $p$-adic result is proved. Let $\pi$ be a supercuspidal representation defined on $G(F)$, where $F$ is a $p$-adic field. Then $\mathbf{a}$ is a cuspidal unipotent orbit for $\pi$ provided there is a representative $u_{\mathbf{a}}$ defined on $G(F)$, which cannot be conjugated inside a Levi part of some proper parabolic subgroup of $G(F)$. It could happen that there is another representative which can be conjugated inside a Levi part of some proper parabolic subgroup of $G(F)$. We expect this result to be true for automorphic representations. However, we will state this result somewhat differently.

The situation for the group $G = GL_n$ is clear. It is well known that every cuspidal representation defined on $GL_n(\mathbf{A})$ must be generic. Thus, in this case we have $\mathcal{O}_G(\pi) = (n)$. It is also true that this is the only unipotent orbit which cannot be conjugated into a Levi part of $G$.

In [G-R-S2] the structure of $\mathcal{O}_G(\pi)$ was studied for the symplectic group. We start with the general conjecture in this case

CONJECTURE 4.1: *Suppose that $\pi$ is a cuspidal representation defined on the group $Sp_{2n}(\mathbf{A})$. If $\mathbf{a} \in \mathcal{O}_G(\pi)$ then $\mathbf{a}$ consists of even numbers only.*

For example, if $G = Sp_6$, then the only possible cuspidal partitions are $(6), (42)$ and $(2^3)$. It is not hard to prove the conjecture for low rank symplectic groups. In general, the following holds.

THEOREM 4.2 ([G-R-S2] Theorem 2.7): *Suppose that $G = Sp_{2n}$. Then there exists a unipotent orbit $\mathbf{a} \in \mathcal{O}_G(\pi)$ such that $\mathbf{a}$ consists of even numbers only.*

Recall that for some unipotent orbits $\mathbf{a}$ the corresponding set of Fourier coefficients is infinite. This is indeed the case, for example, when $G = Sp_{2n}$ and $\mathbf{a}$ consists of even numbers only. However, it is possible that some of the representatives can be conjugated inside a Levi part of a parabolic and some not. We thus expect that if $\mathbf{a} \in \mathcal{O}_G(\pi)$ and if the representative we choose can be conjugated inside a Levi part, then the corresponding Fourier coefficient will be zero for every choice of data. Consider, for example, the unipotent orbit $\mathbf{a} = (2^2)$ when $G = Sp_4$, that is, assume that $\pi$ is such that $\mathcal{O}_G(\pi) = (2^2)$. In section 2 we wrote down the set of Fourier coefficients which corresponds to this unipotent orbit. The set of all representatives inside $Sp_4(F)$ corresponding to the unipotent orbit $\mathbf{a} = (2^2)$ is given by the set $u(\beta, \gamma)$ defined by

$$u(\beta,\gamma) = \begin{pmatrix} 1 & & & \beta \\ & 1 & \gamma & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad u_0 = \begin{pmatrix} 1 & & 1 & \\ & 1 & & 1 \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

Here $\beta$ and $\gamma$ are defined up to a square class. As one can verify, in the case when $\beta\gamma = -\epsilon^2$ the matrix $u(\beta,\gamma)$ is conjugated inside $Sp_4(F)$ to the matrix $u_0$. One can also verify that only when $\beta\gamma = -\epsilon^2$, the matrix $u(\beta,\gamma)$ can be conjugated in $Sp_4(F)$, inside a Levi part of a parabolic subgroup of $Sp_4$.

In this case, when $\beta\gamma = -\epsilon^2$ and $\mathcal{O}_G(\pi) = (2^2)$, integral (3) is nonzero for some choice of data, if and only if the integral

$$\mathcal{F}_{\mathbf{a}}(\varphi)(g) = \int_{(F\backslash\mathbf{A})^3} \varphi\left(\begin{pmatrix} 1 & & x & y \\ & 1 & z & x \\ & & 1 & \\ & & & 1 \end{pmatrix} g\right)\psi(x)dxdydz \tag{5}$$

is nonzero for some choice of data. Applying some Fourier expansions, one can show that integral (5) can be expressed as a sum of Whittaker coefficients of $\pi$ and a constant term along a unipotent radical of a maximal parabolic. By our assumption that $\mathcal{O}_G(\pi) = (2^2)$, all the Whittaker coefficients, which correspond to the unipotent orbit (4), are zero. Since $\pi$ is cuspidal the constant term is zero. This means that integral (5) is zero for every choice of data. Notice that when $\beta\gamma = -\epsilon^2$ then the stabilizer $C(F)$ contains the split group $GL_1(F)$.

We can generalize this example and conjecture

CONJECTURE 4.3: *Let $\pi$ be a cuspidal representation defined over the group $G(\mathbf{A})$. Let $\mathbf{a} \in \mathcal{O}_G(\pi)$ and assume that all numbers in $\mathbf{a}$ are even. Let $u_{\mathbf{a}}$ denote any representative of the unipotent orbit $\mathbf{a}$ inside the group $G(F)$. Denote by $C_{u_{\mathbf{a}}}(F)$ the stabilizer inside $L(F)$ of the element $u_{\mathbf{a}}$ and by $\mathcal{F}_{u_{\mathbf{a}}}(\varphi)$ the corresponding Fourier coefficient. Then $\mathcal{F}_{u_{\mathbf{a}}}(\varphi)$ is zero for every choice of data if the group $C_{u_{\mathbf{a}}}(\mathbf{A})$ contains some unipotent subgroup.*

It is a natural question whether, given a unipotent orbit $\mathbf{a}$ which consists of even numbers only, there is a cuspidal representation $\pi$ such that $\mathbf{a} \in \mathcal{O}_G(\pi)$? It seems that this is indeed the case. For example, in [I] there is a construction of cuspidal representations $\pi$ for $Sp_{4n}(\mathbf{A})$ such that $\mathcal{O}_G(\pi) = (2^{2n})$. This is of course the smallest cuspidal unipotent orbit for $Sp_{4n}$. In [G-G] it is shown that these representations are precisely the ones that lift to $SO_{8n}(\mathbf{A})$ under the theta correspondence. Hence we can expect that such representations exist for all symplectic groups.

In [G] there are examples of cuspidal representations $\pi$ defined such that $\mathcal{O}_G(\pi)$ contains other partitions which consists of even numbers. For example, for the group $Sp_6(\mathbf{A})$ the construction in [G] implies the existence of cuspidal representations $\pi$ such that $\mathcal{O}_G(\pi) = (42)$.

We also want to state the analogues of Conjecture 4.1 for orthogonal groups. We will do it for even rank groups, that is for $SO_{2n}$. Let $\mathbf{a}$ be a unipotent orbit corresponding to the group $SO_{2n}$. Since even numbers occur with even multiplicities, it thus follows that the total number of odd numbers must be even. This means that we can write $\mathbf{a} = (a_1 \ldots a_r b_1 \ldots b_r)$ where $a_i \geq a_{i+1}$ and $a_i \geq b_j$ for all $1 \leq i, j \leq r$. Based on many cases we checked, we have

CONJECTURE 4.4: *Let $\pi$ be a cuspidal representation defined over the group $SO_{2n}(\mathbf{A})$. Suppose that $\mathbf{a} = (a_1 \ldots a_r b_1 \ldots b_r)$ is a unipotent orbit in $\mathcal{O}_G(\pi)$ where $G = SO_{2n}$. Then all $a_i$ and $b_j$ are odd numbers and $a_r > b_1$.*

For example, $SO_8$ has three unipotent orbits of this type. They are $(71), (53)$ and $(3^2 1^2)$. As for the symplectic groups, it is not hard to prove the conjecture for low rank groups.

Finally, in the exceptional groups it is also expected that only part of the unipotent orbits will be in the set $\mathcal{O}_G(\pi)$ when $\pi$ is a cuspidal representation. For the exceptional group $G_2$ the only two that have this property are the two special unipotent orbits, the orbits $G_2$ and $G_2(a_1)$. We do not know the answer for the other exceptional groups.

## 5. On the set $\mathcal{O}_G(\pi)$ for Eisenstein series

In this section we will give some conjectures regarding the set $\mathcal{O}_G(\pi)$, where $\pi$ is an Eisenstein series or a residue of an Eisenstein series. The idea is to try to relate the structure of the set $\mathcal{O}_M(\tau)$ with the structure of the set $\mathcal{O}_G(\pi)$. Here $M$ is a Levi subgroup of a parabolic subgroup of $G$ and $\tau$ an automorphic representation defined on the group $M(\mathbf{A})$. By definition, the Eisenstein series depends on a choice of a set of complex numbers and so it is expected that the answer will depend on that choice. We start by fixing some notation.

Let $G$ be a split reductive group and let $P$ denote a parabolic subgroup of $G$ with Levi part $M$. Let $\tau$ denote an automorphic representation defined on the group $M(\mathbf{A})$. Let $E_\tau(g, \bar{s})$ denote the Eisenstein series defined on $G(\mathbf{A})$ corresponding to the induced representation $Ind_{P(\mathbf{A})}^{G(\mathbf{A})} \tau \delta_P^{\bar{s}}$. Here $\bar{s} = (s_1, \ldots, s_t)$ denotes a multi-complex variable. Suppose that $M = H_1 \times \cdots \times H_r$, where $H_i$ are some reductive groups. Let $\mathbf{b}_i$ denote a unipotent orbit for the group $H_i$. With this notation, we shall denote by $\mathbf{a} = (\mathbf{b}_1, \ldots, \mathbf{b}_r)$ a unipotent orbit for the group $M$.

5.1 THE SET $\mathcal{O}_G(\pi)$ FOR $G = GL_n$. It will be convenient to start with the group $G = GL_n$. In this case we have $M = GL_{n_1} \times \cdots \times GL_{n_r}$. Let us show

how to associate with each unipotent orbit $\mathbf{a}$ an Eisenstein series. Assume that $\mathbf{a} = (k_1 \ldots k_r)$ and suppose that $\mathbf{a}^t = (m_1 \ldots m_l)$. Let $P_{m_1,\ldots,m_l}$ denote the parabolic subgroup of $GL_n$ whose Levi part is $GL_{m_1} \times \cdots \times GL_{m_l}$. We shall denote by $E(g, \bar{s})$ the Eisenstein series which corresponds to the induced representation $Ind_{P_{m_1,\ldots,m_l}(\mathbf{A})}^{G(\mathbf{A})} \delta_P^{\bar{s}}$. This is a special case where we take $\tau$ to be the trivial representation. We have

CONJECTURE 5.1: *With the above notation, suppose that $Re(s_i)$ is large. We then have $\mathcal{O}_G(E(g, \bar{s})) = \mathbf{a}$.*

We shall illustrate this conjecture for maximal parabolic subgroups of $G$. We have

PROPOSITION 5.2: *For $r \leq n - r$, let $P_{n-r,r}$ denote the maximal parabolic subgroup of $G = GL_n$ whose Levi part is $GL_{n-r} \times GL_r$. Let $E(g, s)$ denote the Eisenstein series corresponding to the induced representation $Ind_{P_{n-r,r}(\mathbf{A})}^{G(\mathbf{A})} \delta_P^s$. Then for $Re(s)$ large, $\mathcal{O}_G(E(g, s)) = (2^r 1^{n-2r})$.*

*Proof:* We have to prove two things. First, we need to show that given any unipotent orbit $\mathbf{a}$ for the group $G$, which is either bigger than or not related to $(2^r 1^{n-2r})$, then the representation $E(g, s)$ has no nonzero Fourier coefficient which is associated with $\mathbf{a}$. Then we have to show that $E(g, s)$ has a nonzero Fourier coefficient which is associated with $(2^r 1^{n-2r})$.

To prove the first part, let $\mathbf{a} = (k_1 \ldots k_r)$ be any unipotent orbit which is bigger than or not related to $(2^r 1^{n-2r})$. This means that either $k_1 \geq 3$ or that $\mathbf{a} = (2^m 1^{n-2m})$ with $m > r$. Let

$$\mathcal{F}_{\mathbf{a}}(g) = \int_{V(F)\backslash V(\mathbf{A})} E(vg, s) \psi_V(v) dv \tag{6}$$

be the Fourier coefficient associated with the unipotent orbit $\mathbf{a}$ as defined by (1). For $Re(s)$ large we unfold the Eisenstein series. Thus we need to study the space $P_{n-r,r} \backslash GL_n / V$ and check that every representative contributes zero to (6). Every such representative is of the form $wu_w$, where $w$ is a Weyl element and $u_w \in U/V$; recall that $U$ is the maximal unipotent subgroup of $G$. To prove this part it is enough to show that given $wu_w$ as above, we can find an element $v \in V$ such that $\psi_V(v) \neq 1$ and $(wu_w)v(wu_w)^{-1} \in P_{n-r,r}$. This is global, analogous to the Key Lemma 2 in [G-R-S3]. In a similar way to the proof of that lemma, we can show the following. If for a Weyl element we can find $v \in V$ such that $\psi_V(v) \neq 1$ and $wvw^{-1} \in P_{n-r,r}$, then we can find such a $v$ for the element $wu_w$. In other words, we need only consider Weyl elements. Assume first that $\mathbf{a} = (k_1 \dots k_r)$ is such that $k_1 \geq 3$. Then one can check that there are at least two one-dimensional unipotent subgroups of $V$, $x_\alpha(r)$ and $x_\beta(r)$, which do not commute and such that $\psi_V(x_\alpha(r)) \neq 1$ and $\psi_V(x_\beta(r)) \neq 1$. Let $U(P_{n-r,r})$ denote the unipotent radical of $P_{n-r,r}$ and let $U(P_{n-r,r})^-$ denote the transpose of $U(P_{n-r,r})$. Since Weyl elements permute the roots, then for every one parameter unipotent subgroup of $V$ which corresponds to a positive root we have $wvw^{-1} \in P_{n-r,r}$ if and only if $wvw^{-1} \notin U(P_{n-r,r})^-$. Since $U(P_{n-r,r})^-$ is abelian, it follows that if $\alpha$ and $\beta$ are two roots which do not commute then either $wx_\alpha(r)w^{-1} \notin U(P_{n-r,r})^-$ or $wx_\beta(r)w^{-1} \notin U(P_{n-r,r})^-$. But this means that either $wx_\alpha(r)w^{-1} \in P_{n-r,r}$ or $wx_\beta(r)w^{-1} \in P_{n-r,r}$. All this implies that if $k_1 \geq 3$ then integral (6) is zero for each choice of data.

Next assume that $\mathbf{a} = (2^m 1^{n-2m})$ with $m > r$. In this case, the group $V$ in (6) is defined as

$$V = \{v = \begin{pmatrix} I_m & & X \\ & I_{n-2m} & \\ & & I_m \end{pmatrix} : X \in Mat_{m\times m}\}.$$

The character $\psi_V$ is defined as $\psi_V(v) = \psi(trX)$. Once again, when unfolding the Eisenstein series in integral (6) it is not hard to check that all representatives of $P_{n-r,r}\backslash GL_n/V$ give zero contribution.

To complete the proof of the proposition, we need to show that integral (6) is not zero when $V$ corresponds to the unipotent orbit $\mathbf{a} = (2^r 1^{n-2r})$. We unfold the Eisenstein series in integral (6) and, checking the double coset space, we get zero contribution from all representatives except from the Weyl element

$$w = \begin{pmatrix} & & I_r \\ & I_{n-2r} & \\ I_r & & \end{pmatrix}.$$

In this case integral (6) equals

$$\int_{V(\mathbf{A})} f(wvg, s)\psi_V(v)dv. \tag{7}$$

It is clear that this integral is factorizable and for $Re(s)$ large it is nonzero. This completes the proof of the proposition. ∎

Another set of interesting representations are the generalized Speh representations. These representations were studied in [J]. To define them, let $\tau$ denote a *generic* representation defined on the group $GL_m(\mathbf{A})$. Denote by $P_{rm}$ the maximal parabolic subgroup of $G = GL_{rm}$ whose Levi part is $GL_m^r$. Let $E_\tau(g, \bar{s})$

denote the Eisenstein series defined on $GL_{rm}(\mathbf{A})$ corresponding to the induced representation $Ind_{P_{rm}(\mathbf{A})}^{G(\mathbf{A})}(\tau|\cdot|^{s_1}\otimes\cdots\otimes\tau|\cdot|^{s_r})\delta_{P_{rm}}^{1/2}$. It is well known that this Eisenstein series has a simple pole at the point

$$\bar{s_0}=\Big(\frac{r-1}{2},\frac{r-3}{2},\ldots,-\frac{(r-3)}{2},-\frac{(r-1)}{2}\Big).$$

We shall denote this residue representation by $E_\tau(g) = Res_{\bar{s}=\bar{s_0}}E_\tau(g,\bar{s})$ and refer to it as the generalized Speh representation. For these residues we have

PROPOSITION 5.3: *With the above notation, we have $\mathcal{O}_G(E_\tau(g)) = (m^r)$.*

*Proof:* We sketch the details. Let $\mathbf{a} = (k_1\ldots k_r)$ be any unipotent orbit of $G$ which is bigger than or not related to $(m^r)$. Clearly $k_1 > m$. Assume that $E_\tau(g)$ has a nonzero Fourier coefficient which is associated with $\mathbf{a}$. Arguing as in [G-R-S2] Lemma 2.6 we deduce that $E_\tau(g)$ has a nonzero Fourier coefficient which is associated with the unipotent orbit $(k_1 1^{mr-k_1})$. As in [G-R-S1] Theorem 8, it follows that if an automorphic representation $\sigma$ of $GL_n(\mathbf{A})$ has a nonzero Fourier coefficient with respect to the unipotent orbit $(t_1 1^{n-t_1})$, then it has a nonzero Fourier coefficient which is associated with $(t_2 1^{n-t_2})$ for any $t_2 < t_1$. From all this we deduce that if we can show that $E_\tau(g)$ has no nonzero Fourier coefficient which is associated with $((m+1)1^{mr-m-1})$, then it has non nonzero Fourier coefficient associated with any unipotent orbit which is bigger than or not related to $(m^r)$.

The proof is similar to the proof of Key Lemma 2 in [G-R-S3]. We analyze the local unramified component of the residue which we denote by $\pi(\tau)$. Arguing as in [G-R-S3] we deduce that $\pi(\tau)$ is a constituent of the induced representation $Ind_{Q_{rm}}^{G}(\chi_1\otimes\cdots\otimes\chi_m)\delta_{Q_{rm}}^{1/2}$. Here $Q_{rm}$ is the parabolic subgroup of $G$ whose Levi part is $GL_r^m$. The characters $\chi_i$ are certain unramified characters of the group $GL_r$ which depend on the representation $\tau$. Since any global Fourier coefficient induces a corresponding local functional on each of its components, it is enough to prove that the representation $\pi(\tau)$ has no nonzero local functional which corresponds to the unipotent orbit $((m+1)1^{mr-m-1})$. Using the Bruhat theory it is enough to show that each double coset representative $g \in Q_{rm}\backslash G/V$ has the following property. For such $g$ there is a $v \in V$ such that $\psi_V(v) \neq 1$ and $gvg^{-1} \in Q_{rm}$. Here $V$ is the unipotent group which corresponds to the unipotent orbit $((m+1)1^{mr-m-1})$ as defined in (1), and $\psi_V$ is the corresponding local additive character. As in Key Lemma 2 in [G-R-S3] and as we explained in some detail in the proof of Proposition 5.2, it follows that we may restrict ourselves to representatives which are in the Weyl group of $GL_{mr}$. Then arguing again as in the above two references, we also deduce that for Weyl elements we can find a $v$ as above. We omit the details.

Next, we need to show that $E_\tau(g)$ has a nonzero Fourier coefficient which is associated with the unipotent orbit $(m^r)$. To describe the Fourier coefficient corresponding to this unipotent orbit, let $V = V_{rm}$ denote the unipotent radical subgroup of $Q_{rm}$. In terms of matrices, this group consists of all upper triangular matrices of the form

$$V_{rm} = \left\{ v = \begin{pmatrix} I_r & x_1 & * & \dots & * \\ & I_r & x_2 & \dots & * \\ & & \ddots & \ddots & * \\ & & & I_r & x_{m-1} \\ & & & & I_r \end{pmatrix} \right\}.$$

Here $x_i \in Mat_{r\times r}$ and the star indicates arbitrary entries. We define the character $\psi_V$ on this group as $\psi_V(v) = \psi(trx_1 + \cdots + trx_{m-1})$. Thus we have to show that the integral

$$\int_{V(F)\backslash V(\mathbf{A})} E_\tau(vg)\psi_V(v)dv$$

is not zero for some choice of data. The idea of the proof is similar to the proof of Proposition 2 in [G-R-S3] and to the proof of Theorem 1 and Lemmas 1 and 2 in [G-R-S4] pp. 889–898. We shall sketch some of the details. Let $w$ denote the Weyl element of $G$ whose $(km+i, (i-1)r+k+1)$ entry is one and zero elsewhere. Here, $0 \le k \le r-1$ and $1 \le i \le m$. Conjugating in the above integral, by this Weyl element, we obtain

$$\int_{L(F)\backslash L(\mathbf{A})} \int_{U_1(F)\backslash U_1(\mathbf{A})} E_\tau(u_1 lwg)\psi_1(u_1)du_1 dl.$$

Here $U_1$ is a certain upper unipotent matrix, $L$ is a certain lower unipotent matrix (these are exactly the unipotent matrices in $V$ which the Weyl element $w$ conjugates to lower unipotent) and $\psi_1$ is the resulting character from the conjugation on the group $U_1$. Arguing as in the above references, we deduce that the above integral is nonzero for some choice of data, if and only if the integral

$$\int_{U(F)\backslash U(\mathbf{A})} E_\tau(ug)\psi_U(u)du \tag{8}$$

is nonzero for some choice of data. To make this deduction we need to preform certain Fourier expansions and also use the fact, already proved, that the residue has no nonzero Fourier coefficients corresponding to any unipotent orbit greater than or not related to $(m^r)$. In the above integral, the group $U$ is the maximal unipotent subgroup of $G$ and for $u = (u_{i,j}) \in U$ we define

$$\psi_U(u) = \psi\bigg( \sum_{i=1}^{m-1} (x_{i,i+1} + x_{m+i,m+i+1} + \cdots + x_{(r-1)m+i,(r-1)m+i+1}) \bigg).$$

In other words, the above integral is an integration of the residue along the constant term of the parabolic subgroup $P_{rm}$ composed with the Whittaker functional on the Levi part of this parabolic. By our assumption the representation $\tau$ is generic. Hence we deduce that the above integral is nonzero for some choice of data. This completes the proof of the Proposition. ∎

All the examples so far indicate the following

CONJECTURE 5.4: *Let $G = GL_n$ and let $\pi$ be an automorphic representation defined on the group $GL_n(\mathbf{A})$. Then the set $\mathcal{O}_G(\pi)$ is a singleton.*

The above conjecture is true for all unitary representations of $GL_n(\mathbf{A})$. Conjecture 5.1 asserts that all Eisenstein series which are associated with an induced representation from the trivial representation do satisfy Conjecture 5.4. In what follows we shall assume that Conjecture 5.4 holds.

We continue to assume that $G = GL_n$. The next natural problem is to study how the set $\mathcal{O}_G(\pi)$ varies as a function of the complex variable $\bar{s}$, and as a function of the representation defined on the Levi part. To state our conjecture regarding the sets $\mathcal{O}_G(\pi)$, where $\pi$ is an Eisenstein series or any of its residues, we fix some notation. Let $P_{n_1,\ldots,n_r}$ denote the standard parabolic subgroup of $GL_n$ whose Levi part is $M_{n_1,\ldots,n_r} = GL_{n_1} \times \cdots \times GL_{n_r}$. Fix a unipotent orbit $\mathbf{a} = (\mathbf{b}_1, \ldots, \mathbf{b}_r)$ of $M_{n_1,\ldots,n_r}$ where $\mathbf{b}_i$ is a unipotent orbit for the group $GL_{n_i}$. We introduce the following

*Definition 5.5:* (1) With the above notation and assuming Conjecture 5.4, let $\mathcal{O}_M(\mathbf{a})$ denote the set of all automorphic representations $\tau = \tau_1 \otimes \cdots \otimes \tau_r$ defined on $M_{n_1,\ldots,n_r}(\mathbf{A})$ such that $\mathcal{O}_M(\tau) = \mathbf{a}$. Here $\tau_i$ is an automorphic representation of $GL_{n_i}(\mathbf{A})$.

(2) Let $\mathcal{O}_G^{\min}(\mathbf{a}, E_\tau(g, \bar{s}))$ denote the set of smallest unipotent orbits (relative to the standard partial ordering) $\mathbf{c}$ of $GL_n$ such that there exists $\tau \in \mathcal{O}_M(\mathbf{a})$ and some values of $\bar{s}$ such that $\mathcal{O}_G(\pi) = \mathbf{c}$. Here $\pi$ denotes the Eisenstein series $E_\tau(g, \bar{s})$ or any of its residues.

(3) Similarly, let $\mathcal{O}_G^{\max}(\mathbf{a}, E_\tau(g, \bar{s}))$ denote the set of largest unipotent orbits (relative to the standard partial ordering) $\mathbf{c}$ of $GL_n$ such that there exists $\tau \in$

$\mathcal{O}_M(\mathbf{a})$ and some values of $\bar{s}$ such that $\mathcal{O}_G(\pi) = \mathbf{c}$. Here $\pi$ is as in the second part.

*Remark:* A-priori, the sets $\mathcal{O}_G^{\min}(\mathbf{a}, E_\tau(g, \bar{s}))$ and $\mathcal{O}_G^{\max}(\mathbf{a}, E_\tau(g, \bar{s}))$ may contain more than one unipotent orbit. However, in all examples known to me, these sets consist of one orbit. Henceforth we shall make this assumption.

We consider a few examples.

*Examples:* (1) Let $G = GL_{rm}$ and let $P_{rm}$ denote the parabolic subgroup of $G$ whose Levi part is $M_{rm} = GL_m^r$. For $1 \le i \le r$, let $\tau_i$ denote any *generic* automorphic representation defined on the group $GL_m(\mathbf{A})$. We denote by $\tau = \tau_1 \otimes \cdots \otimes \tau_r$ the corresponding automorphic representation defined on the group $M_{rm}(\mathbf{A})$. Let $E_\tau(g, \bar{s})$ denote the Eisenstein series defined on the group $G(\mathbf{A})$ corresponding to the induced representation $Ind_{P_{rm}(\mathbf{A})}^{G(\mathbf{A})} \tau \delta_{P_{rm}}^{\bar{s}}$. Since each $\tau_i$ is generic, it follows that $\mathbf{b}_i = (m)$ for all $i$ and hence $\mathbf{a} = (m, \ldots, m)$ is the corresponding unipotent orbit for the group $M_{rm}$. It is well known that if $Re(\bar{s})$ is large then the representation $E_\tau(g, \bar{s})$ is generic. Hence we obtain that $\mathcal{O}_G^{\max}(\mathbf{a}, E_\tau(g, \bar{s})) = (mr)$. To compute $\mathcal{O}_G^{\min}(\mathbf{a}, E_\tau(g, \bar{s}))$, we have to look for a representation $\tau$ and values of $\bar{s}$ such that the unipotent orbit $\mathcal{O}_G(\pi)$ will be the smallest possible. Here $\pi$ denotes the Eisenstein series $E_\tau(g, \bar{s})$ or any of its residues. This will happen if we choose all the $\tau_i$ to be equal, and $\bar{s}$ as defined right before Proposition 5.3. In this case we will obtain the Speh representation. It follows from Proposition 5.3 that $\mathcal{O}_G(\pi) = (m^r)$ in this case. Thus, assuming Conjecture 5.4, we have $\mathcal{O}_G^{\min}(\mathbf{a}, E_\tau(g, s)) \le (m^r)$. To prove an equality we refer to the proof of Proposition 5.3. Even though it is stated only for the residue, one can apply the same arguments and show that integral (8) is actually nonzero even if we replace the residue by the full Eisenstein series and take $\tau$ to be any generic representation. Moreover, from the discussion before (8) one deduces that this Fourier coefficient corresponds to the unipotent orbit $(m^r)$. From this it follows that $\mathcal{O}_G^{\min}(\mathbf{a}, E_\tau(g, s)) = (m^r)$.

(2) In the notation of Proposition 5.2, let $G = GL_n$ and denote by $E(g, s)$ the Eisenstein series corresponding to the induced representation $Ind_{P_{n-r,r}(\mathbf{A})}^{G(\mathbf{A})} \delta_{P_{n-r,r}}^{s}$. In this case $\mathbf{b}_1 = (1^{n-r})$ and $\mathbf{b}_2 = (1^r)$. Hence $\mathbf{a} = (1^{n-r}, 1^r)$. It follows from Proposition 5.2 that for $Re(s)$ large we have $\mathcal{O}_G(E(g, s)) = (2^r 1^{n-r})$. Hence we obtain $\mathcal{O}_G^{\max}(\mathbf{a}, E(g, s)) = (2^r 1^{n-r})$. On the other hand, we know that this Eisenstein series has the trivial representation as its residue. Hence $\mathcal{O}_G^{\min}(\mathbf{a}, E(g, s)) = (1^n)$.

(3) We generate example (2). Let $G = GL_n$ and let $\mathbf{a} = (1^{n_1}, \ldots, 1^{n_r})$ be the trivial unipotent orbit on the group $M_{n_1,\ldots,n_r} = GL_{n_1} \times \cdots \times GL_{n_r}$.

Without loss of generality, assume that $n_1 \geq n_2 \geq \cdots \geq n_r$. In this case the set $\mathcal{O}_M(\mathbf{a})$ contains only the trivial representation. Let $E(g,\bar{s})$ denote the Eisenstein series defined on the group $GL_n(\mathbf{A})$ corresponding to the induced representation $Ind_{P_{n_1,\ldots,n_r}(\mathbf{A})}^{G(\mathbf{A})}\delta_{P_{n_1,\ldots,n_r}}^{\bar{s}}$. Let $\mathbf{c}$ denote the unipotent orbit for the group $GL_n$ defined by $\mathbf{c} = (n_1n_2\ldots n_r)$. It follows from Conjecture 5.1 that for $Re(\bar{s})$ large we have $\mathcal{O}_G(E(g,\bar{s})) = \mathbf{c}^t$. Hence $\mathcal{O}_G^{\max}(\mathbf{a}, E(g,\bar{s})) = \mathbf{c}^t$. On the other hand, this Eisenstein series has the trivial representation as its residue, hence $\mathcal{O}_G^{\min}(\mathbf{a}, E(g,\bar{s})) = (1^n)$.

From all these examples we deduce the following

CONJECTURE 5.6: *With the above notation, and with the notation of Definition 2.2, we have*

$$\mathcal{O}_G^{\max}(\mathbf{a}, E_\tau(g,\bar{s})) = \mathbf{b}_1 + \mathbf{b}_2 + \cdots + \mathbf{b}_r$$

*and*

$$\mathcal{O}_G^{\min}(\mathbf{a}, E_\tau(g,\bar{s})) = \mathbf{b}_1\mathbf{b}_2\ldots\mathbf{b}_r.$$

One can check that the above examples do satisfy Conjecture 5.6.

*Remark:* It follows from [C-M] that $\mathbf{b}_1 + \mathbf{b}_2 + \cdots + \mathbf{b}_r$ is the induced unipotent orbit from $\mathbf{a}$ denoted in the above reference as $Ind_{\mathcal{P}}^{\mathcal{G}}\mathbf{a}$. Here $\mathcal{P}$ corresponds to the parabolic subgroup $P_{n_1,\ldots,n_r}$.

The following lemma is contained in the proof of Lemma 7.2.5 in [C-M].

LEMMA 5.7: *With the above notation, let* $\mathbf{a} = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_r)$ *denote a unipotent orbit for the group* $M_{n_1,\ldots,n_r} = GL_{n_1} \times \cdots \times GL_{n_r}$. *Denote*

$$\mathbf{a}^t = (\mathbf{b}_1^t, \mathbf{b}_2^t, \ldots, \mathbf{b}_r^t).$$

*Then*

$$(\mathbf{b}_1 + \mathbf{b}_2 + \cdots + \mathbf{b}_r)^{\mathbf{t}} = \mathbf{b}_1^t\mathbf{b}_2^t\ldots\mathbf{b}_r^t.$$

From this we deduce

PROPOSITION 5.8 (The Min-Max Principle): *With the above notation, and assuming Conjecture 5.6, we have*

$$\mathcal{O}_G^{\min}(\mathbf{a}^t, E_\tau(g,\bar{s})) = \mathcal{O}_G^{\max}(\mathbf{a}, E_\tau(g,\bar{s}))^{\mathbf{t}}.$$

*This can also be written as*

$$\mathcal{O}_G^{\max}(\mathbf{a}^t, E_\tau(g,\bar{s})) = \mathcal{O}_G^{\min}(\mathbf{a}, E_\tau(g,\bar{s}))^{\mathbf{t}}.$$

*Example:* Let $G = GL_5$ and let $P_{2,3}$ denote the maximal standard parabolic subgroup of $G$ whose Levi part is $M_{2,3} = GL_2 \times GL_3$. In the following table, assuming Conjecture 5.6 holds, we list all possible values of $\mathcal{O}_G^{\max}(\mathbf{a}, E_\tau(g,s))$ and $\mathcal{O}_G^{\min}(\mathbf{a}, E_\tau(g,s))$.

(9)

| $\mathbf{a}$ | $\mathcal{O}_G^{\min}(\mathbf{a}, E_\tau(g,s)) = \mathbf{b}_1\mathbf{b}_2$ | $\mathcal{O}_G^{\max}(\mathbf{a}, E_\tau(g,s)) = \mathbf{b}_1 + \mathbf{b}_2$ |
|---|---|---|
| $(2,3)$ | $(32)$ | $(5)$ |
| $(2,21)$ | $(2^21)$ | $(41)$ |
| $(2,1^3)$ | $(21^3)$ | $(31^2)$ |
| $(1^2,3)$ | $(31^2)$ | $(41)$ |
| $(1^2,21)$ | $(21^3)$ | $(32)$ |
| $(1^2,1^3)$ | $(1^5)$ | $(2^21)$ |

To demonstrate Proposition 5.8 we write down the list of all unipotent orbits of $GL_5$ according to the partial order mentioned in section 2. We have

$$(5) > (41) > (32) > (31^2) > (2^21) > (21^3) > (1^5) \tag{10}$$

Suppose, for example, that $\mathbf{a} = (2,21)$. Then $\mathbf{b}_1 = (2)$ and $\mathbf{b}_2 = (21)$. Since $\mathbf{b}_1^t = (1^2)$ and $\mathbf{b}_2^t = (21)$ it follows that $\mathbf{a}^t = (1^2,21)$. Hence, from (9) it follows that $\mathcal{O}_G^{\min}(\mathbf{a}^t, E_\tau(g,s)) = (21^3)$. On the other hand, also from (9) we have $\mathcal{O}_G^{\max}(\mathbf{a}, E_\tau(g,s)) = (41)$. From table (10) we see that $(41)^{\mathbf{t}} = (21^3)$ as expected from Proposition 5.8. As another example, let $\mathbf{a} = (2,1^3)$. Then $\mathbf{a}^t = (1^2,3)$ and hence $\mathcal{O}_G^{\min}(\mathbf{a}^t, E_\tau(g,s)) = (31^2)$. From (9) it follows that $\mathcal{O}_G^{\max}(\mathbf{a}, E_\tau(g,s)) = (31^2)$. From (10) it follows that $(31^2)^{\mathbf{t}} = (31^2)$.

To finish this section, let us mention that this is only the starting point. In other words, given an Eisenstein series $E_\tau(g,\bar{s})$ defined on the group $GL_n(\mathbf{A})$, then it can possibly have several residue representations. Proposition 5.8 demonstrates the unipotent orbits corresponding to the Eisenstein series in general position, and to the smallest possible residue. The other residues should correspond to inter-median unipotent orbits. We shall discuss this in some detail in section 6.

5.2 The set $\mathcal{O}_G(\pi)$ for other classical groups. In this section we sketch the conjectures stated for $GL_n$ in the previous sub-section, this time for the other classical groups. Let $G$ denote a split symplectic or orthogonal group of rank $n$. Let $P_{n_1,\ldots,n_r,m}$ denote the standard parabolic subgroup of $G$ whose Levi part is $M_{n_1,\ldots,n_r,m} = GL_{n_1} \times \cdots \times GL_{n_r} \times H_m$, where $H_m$ is a reductive group of the same type of $G$ but with rank $m$ which is smaller than $n$. Let $(\tau,\sigma) = \tau_1 \otimes \cdots \otimes \tau_r \otimes \sigma$ denote an automorphic representation defined on the group $M_{n_1,\ldots,n_r,m}(\mathbf{A})$. Here, each $\tau_i$ is a representation of $GL_{n_i}(\mathbf{A})$ and $\sigma$ is a representation defined on the group $H_m(\mathbf{A})$.

Since no unipotent orbit corresponding to $G$ is special, it follows from Theorem 3.1 that analogous to Conjecture 5.1 for the other classical groups is

CONJECTURE 5.9: *Let* $\mathbf{c}$ *denote a special unipotent orbit for the group* $G$. *Then there exists an automorphic representation* $\pi$, *defined on* $G(\mathbf{A})$ *such that* $\mathcal{O}_G(\pi) = \mathbf{c}$.

It is not hard to verify this conjecture for low rank groups.

Things are also different as regard to Conjecture 5.4. In other words, it is not clear if the set $\mathcal{O}_G(\pi)$ is a singleton, or if there are examples of representations such that there are more than one unipotent orbit in this set. However, we do expect the following to hold.

CONJECTURE 5.10: *Let* $\pi$ *denote an automorphic representation defined on the group* $G(\mathbf{A})$. *Suppose that* $\mathbf{c}_1$ *and* $\mathbf{c}_2$ *are both in* $\mathcal{O}_G(\pi)$. *Then* $dim\mathbf{c}_1 = dim\mathbf{c}_2$. *(For the definition of the dimension of a unipotent orbit, see [C-M] Corollary 6.1.4.)*

In the remainder of these notes we shall make the following

ASSUMPTION 5.11: *Henceforth, we shall restrict our attention to the set of all automorphic representations* $\pi$ *defined on* $G(\mathbf{A})$ *such that* $\mathcal{O}_G(\pi)$ *is a singleton.*

To state our conjecture, let $\mathbf{a} = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_r, \mathbf{d})$ denote a unipotent orbit for the group $M_{n_1,\ldots,n_r,m}$. Here, each $\mathbf{b}_i$ is a unipotent orbit for the group $GL_{n_i}$ and $\mathbf{d}$ is a unipotent orbit for the group $H_m$. As in Definition 5.5, we have

*Definition 5.12:* (1) With the above notation, let $\mathcal{O}_M(\mathbf{a})$ denote the set of all automorphic representations $(\tau, \sigma) = \tau_1 \otimes \cdots \otimes \tau_r \otimes \sigma$ defined on $M_{n_1,\ldots,n_r,m}(\mathbf{A})$ such that $\mathbf{a} \in \mathcal{O}_M(\tau)$. Here, $\tau_i$ is an automorphic representation of $GL_{n_i}(\mathbf{A})$ and $\sigma$ is an automorphic representation of the group $H_m(\mathbf{A})$.

(2) Let $\mathcal{O}_G^{\min}(\mathbf{a}, E_{\tau,\sigma}(g, \bar{s}))$ denote the set of smallest unipotent orbits $\mathbf{c}$ of $G$ such that there exists $(\tau, \sigma) \in \mathcal{O}_M(\mathbf{a})$ and some value of $\bar{s}$ such that $\mathcal{O}_G(\pi) = \mathbf{c}$. Here, $\pi$ denotes the Eisenstein series $E_\tau(g, \bar{s})$ or any of its residues.

(3) Similarly, let $\mathcal{O}_G^{\max}(\mathbf{a}, E_{\tau,\sigma}(g, \bar{s}))$ denote the set of all largest unipotent orbits $\mathbf{c}$ of $G$ such that there exists $(\tau, \sigma) \in \mathcal{O}_M(\mathbf{a})$ and some value of $\bar{s}$ such that $\mathcal{O}_G(\pi) = \mathbf{c}$. Here, $\pi$ is as in the second part.

We emphasize that this definition is valid under Assumption 5.11. However, we do believe that if we remove that assumption one should be able to modify Definition 5.12 accordingly.

*Example:* Let $G = Sp_{4n}$ and let $P_{2n}$ denote the Siegel parabolic subgroup of $G$, that is, the parabolic subgroup whose Levi part is $M = GL_{2n}$. Let $\tau$ denote any generic representation of $GL_{2n}$. Thus, we have $\mathcal{O}_M(\tau) = (2n)$ and $\mathbf{a} = \mathbf{b_1} = (2n)$. Let $E_\tau(g,s)$ denote the Eisenstein series defined on $G(\mathbf{A})$ corresponding to the induced representation $Ind_{P_{2n}(\mathbf{A})}^{G(\mathbf{A})}\tau\delta_{P_{2n}}^s$. It is well known that if $Re(s)$ is large, then $E_\tau(g,s)$ is generic. Hence $\mathcal{O}_G^{\max}(\mathbf{a}, E_\tau(g,s)) = (4n)$. On the other hand, let $\tau$ denote a cuspidal representation such that the exterior square $L$ function has a simple pole. In this case, the Eisenstein series $E_\tau(g,s)$ has a residue at $s=1$ and, if we denote this residue by $E_\tau(g)$, then it follows as in [G-R-S1] that $\mathcal{O}_G(E_\tau(g)) = ((2n)^2)$. Arguing as in [G-R-S1] and in a similar way to Example 1 after Definition 5.5, we deduce that $\mathcal{O}_G^{\min}(\mathbf{a}, E_\tau(g,s)) = ((2n)^2)$.

Now we can state the analogy of Conjecture 5.6. For simplicity, we shall assume that $G$ is either $Sp_{2n}$ or $SO_{2n}$. We have

CONJECTURE 5.13: *With the above notation, we have*

$$\mathcal{O}_G^{\max}(\mathbf{a}, E_{\tau,\sigma}(g,\bar{s})) = ((\mathbf{2b_1} + \mathbf{2b_2} + \cdots + \mathbf{2b_r} + \mathbf{d})_G)^{S(G)},$$
$$\mathcal{O}_G^{\min}(\mathbf{a}, E_{\tau,\sigma}(g,\bar{s})) = (\mathbf{b}_1\mathbf{b}_1\mathbf{b}_2\mathbf{b}_2\ldots\mathbf{b}_r\mathbf{b}_r\mathbf{d})^{S(G)}.$$

*Here,* $\mathbf{2b_i} = \mathbf{b_i} + \mathbf{b_i}$ *and the definitions of* $\mathbf{c}_G$ *and* $\mathbf{c}^{S(G)}$ *are given in section 2.*

Notice that $\mathbf{b}_1 b_1 \mathbf{b}_2 b_2 \ldots \mathbf{b}_r b_r d$ is $G$ admissible if and only if $\mathbf{d}$ is $H_m$ admissible. Hence we don't need to compute the $G$ collapse of $\mathbf{b}_1\mathbf{b}_1\mathbf{b}_2\mathbf{b}_2\ldots\mathbf{b}_r\mathbf{b}_r d$. As in the $GL_n$ case, it follows from [C-M] Theorem 7.3.3 that

$$(\mathbf{2b_1} + \mathbf{2b_2} + \cdots + \mathbf{2b_r} + \mathbf{d})_\mathbf{G}$$

is the induced unipotent orbit corresponding to $\mathbf{a}$ and the parabolic subgroup $P_{n_1,\ldots,n_r,m}$.

*Example:* Let $G = Sp_8$. In the following two tables we shall list the unipotent orbits $((\mathcal{O}_G^{\max}(\mathbf{a})_G)^S$ and $\mathcal{O}_G^{\min}(\mathbf{a})^S$ (we omit reference to the Eisenstein series in the notation), for all four maximal parabolic subgroups of $G$. The first table contains the relevant values for the Eisenstein series $E_{\tau,\sigma}(g,s)$ of two maximal parabolic subgroups. The left hand side corresponds to the maximal parabolic subgroup whose Levi part is $GL_4$, and the right hand side to the parabolic subgroup whose Levi part is $GL_2 \times Sp_4$.

(11)

| $\mathbf{a}$ | $\mathcal{O}_G^{\min}(\mathbf{a})^S$ | $(\mathcal{O}_G^{\max}(\mathbf{a})_G)^S$ |
|---|---|---|
| $(4)$ | $(44)$ | $(8)$ |
| $(31)$ | $(3^21^2)$ | $(62)$ |
| $(2^2)$ | $(2^4)$ | $(4^2)$ |
| $(21^1)$ | $(2^21^4)$ | $(42^2)$ |
| $(1^4)$ | $(1^8)$ | $(2^4)$ |

| $\mathbf{a}$ | $\mathcal{O}_G^{\min}(\mathbf{a})^S$ | $(\mathcal{O}_G^{\max}(\mathbf{a})_G)^S$ |
|---|---|---|
| $(2,4)$ | $(42^2)$ | $(8)$ |
| $(2,2^2)$ | $(2^4)$ | $(62)$ |
| $(2,1^4)$ | $(2^21^4)$ | $(421^2)$ |
| $(1^2,4)$ | $(421^2)$ | $(62)$ |
| $(1^2,2^2)$ | $(2^21^4)$ | $(4^2)$ |
| $(1^2,1^4)$ | $(1^8)$ | $(3^21^2)$ |

The left hand side of the following table corresponds to the parabolic subgroup whose Levi part is $GL_3 \times SL_2$, and the right hand side to the parabolic subgroup whose Levi part is $GL_1 \times Sp_6$.

(12)

| $\mathbf{a}$ | $\mathcal{O}_G^{\min}(\mathbf{a})^S$ | $(\mathcal{O}_G^{\max}(\mathbf{a})_G)^S$ |
|---|---|---|
| $(3,2)$ | $(3^22)$ | $(8)$ |
| $(21,2)$ | $(2^4)$ | $(62)$ |
| $(1^3,2)$ | $(2^21^4)$ | $(42^2)$ |
| $(3,1^2)$ | $(3^21^2)$ | $(62)$ |
| $(21,1^2)$ | $(2^21^4)$ | $(4^2)$ |
| $(1^3,1^2)$ | $(1^8)$ | $(3^22)$ |

| $\mathbf{a}$ | $\mathcal{O}_G^{\min}(\mathbf{a})^S$ | $(\mathcal{O}_G^{\max}(\mathbf{a})_G)^S$ |
|---|---|---|
| $(1,6)$ | $(62)$ | $(8)$ |
| $(1,42)$ | $(421^2)$ | $(62)$ |
| $(1,3^2)$ | $(3^21^2)$ | $(4^2)$ |
| $(1,2^3)$ | $(2^4)$ | $(42^2)$ |
| $(1,2^21^2)$ | $(2^21^4)$ | $(421^2)$ |
| $(1,1^6)$ | $(1^8)$ | $(2^21^4)$ |

Notice that we did not include the unipotent orbits which correspond to non-special orbits. For example, on the right hand side of table (11), we did not include the case where $\mathbf{a} = (2,21^2)$. That is because the unipotent orbit $(21^2)$ is non-special for the group $Sp_4$.

As an example for the construction of the above tables, consider the entries corresponding to $\mathbf{a} = (1^2,4)$ on the right hand side of table (11). In this case $\mathbf{b}_1 = (1^2)$ and $\mathbf{d} = (4)$. We have $\mathbf{b}_1\mathbf{b}_1\mathbf{d} = (41^4)$. This partition is $Sp_8$ admissible but not special. Since $(41^4)^S = (421^2)$, this is the value corresponding to $\mathcal{O}_G^{\min}(\mathbf{a})^S$. We also have $\mathbf{2b_1} + \mathbf{d} = (\mathbf{62})$, which is $Sp_8$ admissible and special. Hence, this is the value for $(\mathcal{O}_G^{\max}(\mathbf{a})_G)^S$. As another example, consider the entry $\mathbf{a} = (1,3^2)$ on the right hand side of table (12). In this case, $\mathbf{b}_1\mathbf{b}_1\mathbf{d} = (3^21^2)$ and $\mathbf{2b_1} + \mathbf{d} = (\mathbf{53})$. Since $(53)_G = (4^2)$, we obtain the indicated values for this case.

To consider the analogy of Proposition 5.8 to the classical groups, let us first consider the example of $Sp_8$. We list all the special unipotent orbits for this group according to the partial order in this case. We have

$$
\begin{array}{ccccc}
 & & (8) & & \\
 & & | & & \\
 & & (62) & & \\
 & & | & & \\
 & & (4^2) & & \\
 & & | & & \\
 & & (42^2) & & \\
 & / & & \backslash & \\
(421^2) & & & & (3^22) \\
 & \backslash & & / & \\
 & & (3^21^2) & & \\
 & & | & & \\
 & & (2^4) & & \\
 & & | & & \\
 & & (2^21^4) & & \\
 & & | & & \\
 & & (1^8) & &
\end{array}
\tag{13}
$$

As can be seen all the above examples satisfy the following

PROPOSITION 5.14 (The Min-Max Principle): *With the above notation and assuming Conjecture 5.13, we have*

$$\mathcal{O}_G^{\min}(\mathbf{a}^t, E_\tau(g,\bar{s})) = \mathcal{O}_G^{\max}(\mathbf{a}, E_\tau(g,s))^{\mathbf{t}}.$$

*This can also be written as*

$$\mathcal{O}_G^{\max}(\mathbf{a}^t, E_\tau(g,\bar{s})) = \mathcal{O}_G^{\min}(\mathbf{a}, E_\tau(g,s))^{\mathbf{t}}.$$

5.3 THE SET $\mathcal{O}_G(\pi)$ FOR THE EXCEPTIONAL GROUPS. In the case when $G$ is an exceptional group we expect a similar situation as in the classical groups. However, it is not so clear what are the precise statements. We do have some indications that an analogous Min-Max principle should hold. We shall vaguely give some details.

It clearly holds for the exceptional group $G_2$. In this case we have

(14)

| $\mathbf{a}$ | $\mathcal{O}_G^{\min}(\mathbf{a})^S$ | $\mathcal{O}_G^{\max}(\mathbf{a})^S$ | |
|---|---|---|---|
| (2) | $G_2(a_1)$ | $G_2$ | |
| $(1^2)$ | 1 | $G_2(a_1)$ | |

$$
\begin{array}{c}
G_2 \\
| \\
G_2(a_1) \\
|
\end{array}
$$

To explain (14), we recall that $G_2$ has two maximal parabolic subgroups and three special unipotent orbits. The three unipotent orbits and their corresponding ordering are listed on the rightmost column. The first three columns are the minimal and maximal orbits corresponding to the Eisenstein series corresponding to the induced representations from the maximal parabolic subgroups of $G_2$. These two parabolic subgroups have the Levi part $GL_2$.

5.4 A DIMENSION FORMULA. In this section, we would like to prove a simple formula regarding the dimension of the set $\mathcal{O}_G^{\max}(\mathbf{a}, E_\tau(g,\bar{s}))$. Let $\pi$ be an automorphic representation defined on the classical group $G(\mathbf{A})$, and assume that $\mathcal{O}_G(\pi) = \mathbf{c}$. (In particular, we assume Conjecture 5.4.) In [C-M], the notion of the dimension of a unipotent orbit is defined. In [C-M] Corollary 6.1.4, they compute the dimension of these orbits. Following [K] p. 158 remark 3 and the references given there, we introduce the following

*Definition 5.15:* With the above notation we define the Gelfand–Kirillov dimension of $\pi$ to be $\dim \mathbf{c}/2$, and denote this number by $\dim \pi$.

Let $P$ denote a parabolic subgroup of $G$ whose Levi part is $M$ and with unipotent radical $U(P)$. Let $\tau$ denote an automorphic representation defined on the group $M(\mathbf{A})$. We shall assume that $\mathcal{O}_M(\tau) = \mathbf{a}$ is a singleton. Suppose that $M = M_1 \times \cdots \times M_r$ and assume that $\tau = \tau_1 \otimes \cdots \otimes \tau_r$. In this case we define $\dim \tau = \dim \tau_1 + \cdots + \dim \tau_r$.

The following proposition is a trivial consequence of Conjectures 5.6 and 5.13. We include it to emphasize the relation between the Gelfand–Kirillov dimensions of the representations in question and the dimension of the group $U(P)$. It would be nice if a similar relation could be found for $\dim \mathcal{O}_G^{\min}(\mathbf{a}, E_\tau(g,\bar{s}))$.

PROPOSITION 5.16: *With the above notation and assuming Conjectures 5.6 and 5.13, we have the identity*

$$\dim \mathcal{O}_G^{\max}(\mathbf{a}, E_\tau(g,\bar{s})) = \dim \tau + \dim U(P).$$

*Proof:* We shall give a proof for the case $G = GL_n$. In this case, we can identify unipotent orbits with partitions of $n$. Let $\lambda = (k_1 k_2 \ldots k_p)$ with $k_i \geq k_{i+1}$ be a given partition. Then the dimension of $\lambda$ is given by $\dim \lambda = n^2 - \sum_{i=1}^{p} s_i^2$ where $s_i = |\{j : d_j \geq i\}|$. A simple computation implies that

$$\dim \lambda = n^2 - (k_1 + 3k_2 + 5k_3 + \cdots + (2p-1)k_p).$$

Denote by $S_\lambda$ the sum in the parentheses on the right hand side of the above equality.

For the group $GL_n$ we have $M = GL_{n_1} \times \cdots \times GL_{n_r}$. Also, if $\mathbf{a} = (\mathbf{b}_1, \ldots, \mathbf{b}_r)$ we shall denote, for all $1 \le i \le r$, by $\lambda_i$ the partition corresponding to the unipotent orbit $\mathbf{b}_i$. Thus we have

$$\dim \tau = \sum_{i=1}^{r} \dim \tau_i = \frac{1}{2}\sum_{i=1}^{r}(r_i^2 - S_{\lambda_i}).$$

Also, we have $dimU(P) = \sum_{1\le i<j\le r} n_i n_j$. From Conjecture 5.6 we have $\mathcal{O}_G^{\max}(\mathbf{a}, E_\tau(g,\bar{s})) = \mathbf{b}_1 + \cdots + \mathbf{b}_r$. This implies that

$$\dim \mathcal{O}_G^{\max}(\mathbf{a}, E_\tau(g,\bar{s})) = \frac{1}{2}((n_1 + \cdots + n_r)^2 - S_{\lambda_1+\cdots+\lambda_r}).$$

Hence to prove the Proposition it is enough to show that $S_{\lambda_1} + \cdots + S_{\lambda_r} = S_{\lambda_1+\cdots+\lambda_r}$. This is verified easily from the definition of $\lambda_1 + \cdots + \lambda_r$. ∎

## 6. The graph of an Eisenstein series

In section 5 we have established a conjectural formula for the sets $\mathcal{O}_G^{\max}(\mathbf{a},\pi)$ and $\mathcal{O}_G^{\min}(\mathbf{a},\pi)$, where $\pi$ is an Eisenstein series and $\mathbf{a}$ is a unipotent orbit corresponding to the Levi part of the parabolic subgroup from which we induce. Since an Eisenstein series can have several residues at various points, it is natural to consider the sets $\mathcal{O}(\pi')$ where $\pi'$ is a residue of the Eisenstein series $\pi$. To avoid problems of intertwining operators, we shall only consider those residues which correspond to points in the positive Weyl chamber. In fact, the set $\mathcal{O}_G^{\min}(\mathbf{a},\pi)$ is conjectural for the set which corresponds to a certain residue of $\pi$. To make things clearer, let us start with a simple example from the group $G = GL_n$.

Let $n = m + k$ and assume that $m \ge k$. Let $P_{m,k}$ denote the standard maximal parabolic subgroup of $GL_n$ whose Levi part is $GL_m \times GL_k$. Let $E(g,s)$ denote the Eisenstein series defined on the group $GL_n(\mathbf{A})$ which corresponds to the induced representation $Ind_{P_{m,k}(\mathbf{A})}^{GL_n(\mathbf{A})}\delta_{P_{m,k}}^s$. It follows from Proposition 5.2 that $\mathcal{O}_G^{\max}(\mathbf{a}, E(g,s)) = (2^k 1^{m-k})$ where $\mathbf{a} = (1^m, 1^k)$. Since the identity representation is clearly a residue of this Eisenstein series, it thus follows that $\mathcal{O}_G^{\min}(\mathbf{a}, E(g,s)) = (1^{m+k})$.

This Eisenstein series can have more residual representations. In fact, it is not hard to check that the poles of $E(g,s)$ are determined by the poles of $\prod_{i=1}^{k}\zeta(ns-(n-i))$. We consider those residues which occur when $Re(s) > 1/2$. This last product has simple poles at $s_i = (n-i+1)/n$ where $1 \le i \le k$. Let us denote $\theta_i(g) = Res_{s=s_i}E(g,s)$. In a similar way to [G-S] (see also [W] for the local version) one can show that $\mathcal{O}(\theta_i) = (2^i 1^{m+k-2i})$. We can state this result as follows. Given any unipotent orbit $\lambda$ such that $\mathcal{O}_G^{\min}(\mathbf{a}, E(g,s)) \le \lambda \le \mathcal{O}_G^{\max}(\mathbf{a}, E(g,s))$, there exists a residual representation $\theta$ of $E(g,s)$ such that $\mathcal{O}_G(\theta) = \lambda$.

It is natural to introduce the following

*Definition 6.1:* Let $E_\tau(g,\bar{s})$ denote an Eisenstein series defined on a reductive group $G(\mathbf{A})$ associated with the induced representation $Ind_{P(\mathbf{A})}^{G(\mathbf{A})}\tau\delta_P^{\bar{s}}$. Here, $P$ is a parabolic subgroup of the group $G$, and $\tau$ is an automorphic representation defined on the Levi part of $P$. Let $\pi$ denote $E_\tau(g,\bar{s})$ or any one of its residues. Denote by $\Gamma_G(E_\tau(g,\bar{s}))$ the set of all unipotent orbits $\lambda$ for the group $G$, such that there exists $\pi$ as above with $\mathcal{O}_G(\pi) = \lambda$. We call $\Gamma_G(E_\tau(g,\bar{s}))$ the graph of the Eisenstein series $E_\tau(g,\bar{s})$. (Let us remark that in this definition we assumed that $\mathcal{O}(\pi)$ is a singleton.)

In the above example for the group $GL_{m+k}$ it follows that $\Gamma_G(E(g,s)) = \{(2^i 1^{m+k-2i}) : 1 \le i \le k\}$. As another example, let $E(g,\bar{s})$ denote the Eisenstein series defined on $GL_n(\mathbf{A})$ associated with the induced representation $Ind_{B(\mathbf{A})}^{GL_n(\mathbf{A})}\delta_B^{\bar{s}}$. Here $B$ is the Borel subgroup of $GL_n$. It is not hard to show that any degenerate Eisenstein series defined on $GL_n(\mathbf{A})$ is a residue of $E(g,\bar{s})$. In view of Conjecture 5.1, one expects that $\Gamma_G(E(g,\bar{s}))$ will contain every unipotent orbit of the group $GL_n$.

It is clear that if $\lambda \in \Gamma_G(E_\tau(g,\bar{s}))$ then $\lambda \le \mathcal{O}_G^{\max}(\mathbf{a}, E_\tau(g,\bar{s}))$. It is not clear that $\mathcal{O}_G^{\min}(\mathbf{a}, E_\tau(g,\bar{s})) \le \lambda$; however, we expect this to be true. Clearly this is true if $E_\tau(g,\bar{s})$ is a degenerate Eisenstein series, since then $\mathcal{O}_G^{\min}(\mathbf{a}, E_\tau(g,\bar{s}))$ corresponds to the smallest unipotent orbit of the group $G$. It is possible that $\lambda = \mathcal{O}_G^{\max}(\mathbf{a}, E_\tau(g,\bar{s}))$ and that $\lambda = \mathcal{O}_G(\pi)$ where $\pi$ is a residue of $E_\tau(g,\bar{s})$. This can be seen from the following example. Let $G = Sp_4$, and let $E(g,s)$ denote the Eisenstein series defined on $G(\mathbf{A})$ corresponding to the induced representation $Ind_{P(\mathbf{A})}^{G(\mathbf{A})}\delta_P^s$. Here $P$ is the standard maximal parabolic subgroup of $G$ whose Levi part is $GL_2$. Clearly we have $\mathcal{O}_G^{\max}((\mathbf{1^2}), E(g,s)) = (2^2)$ and $\mathcal{O}_G^{\min}((\mathbf{1^2}), E(g,s)) = (1^4)$. One can easily check that this Eisenstein series has two residual representations. First we have the identity representation, and there is also another nontrivial one, which we denote by $\pi$. Since $(1^4)$ and $(2^2)$ are the only special unipotent orbits which are less than or equal to $(2^2)$ (the unipotent orbit $(21^2)$ is not special), it thus follows that $\mathcal{O}_G(\pi) = (2^2) = \mathcal{O}_G^{\max}((\mathbf{1^2}), E(g,s))$.

At this point, we are not in a position to state some conjectures regarding the structure of a graph of an Eisenstein series. However, from the above examples and others, some similarities occur which we would like to point out. To make things clearer, let us focus on Eisenstein series corresponding to induction from maximal parabolic subgroups. In other words, let $E_\tau(g,s)$ denote an Eisenstein series corresponding to an induction from the automorphic representation $\tau$ which is defined on the Levi part of a maximal parabolic subgroup of $G$. Assume that all poles of $E_\tau(g,s)$ are simple. For $1 \le i \le n$, denote by $\pi_i$ the residues of this Eisenstein series at the point $s_i$ where we assume that $Re(s_i) < Re(s_{i+1})$. Let $\mathcal{O}(\pi_i) = \lambda_i$. (We assume that $\mathcal{O}(\pi_i)$ are all singletons.) Based on the above examples, it is natural to ask the following questions.

(1) Is it true that $\mathcal{O}_G^{\min}(\mathbf{a}, E_\tau(g,s)) \le \lambda_n \le \ldots \le \lambda_1 \le \mathcal{O}_G^{\max}(\mathbf{a}, E(g,s))$?

(2) Consider the set of all unipotent orbits of $G$ as a graph. By that we mean the following. Let $\mu$ and $\lambda$ be two unipotent orbits and assume that $\mu < \lambda$. We then connect these two partitions by an edge, if there is no unipotent orbit $\nu$ such that $\mu < \nu < \lambda$. This way we may view the graph $\Gamma_G(E_\tau(g,s))$ as a subgraph of the graph which consists of all unipotent orbits. Is it true that $\Gamma_G(E_\tau(g,s))$ is a connected subgraph? There is of course the problem of how to treat the non-special unipotent orbits, if they exist. As follows from Theorem 3.1, if $\lambda$ is a unipotent orbit which is not special, then there are no representations $\pi$ defined on the group $G(\mathbf{A})$ such that $\mathcal{O}_G(\pi) = \lambda$. When we ask if $\Gamma_G(E_\tau(g,s))$ is connected, it is not clear if one should ignore the non-special orbits or maybe adjust the definition of the graph in a suitable way. See the next section for some more explanation.

## 7. Some examples

In this section we will give some examples for Conjecture 5.6, and an example related to the questions posed in Section 6 related to the graph of an Eisenstein series. We start with

(1) Let $G = GL_3$. Let $\tau$ denote an automorphic representation defined on the group $GL_2(\mathbf{A})$. Let $P$ denote one of the maximal parabolic subgroups of $G$. Its Levi part is $GL_2 \times GL_1$. Let $E_\tau(g,s)$ denote the Eisenstein series defined on $GL_3(\mathbf{A})$ corresponding to the induced representation $Ind_{P(\mathbf{A})}^{G(\mathbf{A})}\tau\delta_P^s$. The relevant table for this case is

(15)

| $\mathbf{a}$ | $\mathcal{O}_G^{\min}(\mathbf{a})$ | $\mathcal{O}_G^{\max}(\mathbf{a})$ |
|---|---|---|
| $(2,1)$ | $(21)$ | $(3)$ |
| $(1^2,1)$ | $(1^3)$ | $(21)$ |

The second row is clear. It corresponds to the case when $\tau$ is the trivial representation on $GL_2(\mathbf{A})$. Clearly, the Eisenstein series is not generic for all values of

$s$, and it is also clear that one can obtain the identity representation on $GL_3(\mathbf{A})$ as a residue of this Eisenstein series.

The first row of (15) corresponds to the case when $\tau$ is generic. If we take $Re(s)$ large, then it is well known that $E_\tau(g,s)$ is generic. Hence the rightmost entry in the first row. Since $\tau$ is generic, it is either a cuspidal representation or an Eisenstein series. In this case we cannot obtain the identity representation of $GL_3(\mathbf{A})$ as a residue of $E_\tau(g,s)$. Hence $\mathcal{O}_G^{\min}((2,1))$ is at least the unipotent orbit (21).

Thus, to verify the second entry on the first row, we need to find a generic representation $\tau$ and a certain value of $s$ such that $\mathcal{O}_G(\pi) = (21)$, where $\pi$ is the above Eisenstein series or any of its residues. In this case we will look for a residue.

Let $\tau = E(h,\nu)$ denote the Eisenstein series defined on the group $GL_2(\mathbf{A})$ corresponding to the induced representation $Ind_{B(\mathbf{A})}^{GL_2(\mathbf{A})}\delta_{B_2}^{\nu}$. The poles of the Eisenstein series $E_\tau(g,s)$ are determined by its constant terms. It is not hard to check that the poles of this Eisenstein series are determined by

$$\frac{\zeta(3s+\nu-2)\zeta(3s-\nu-1)}{\zeta(3s+\nu-1)\zeta(3s-\nu)}.$$

For $Re(\nu)$ large, if we choose $s = (\nu+2)/3$ then $E_\tau(g,s)$ has a simple pole. Denote this residue by $E_\nu(g) = Res_{s=(\nu+2)/3}E_\tau(g,s)$. To show that this residue representation is not generic, we proceed as follows. Consider a non-archimedean unramified place for the Eisenstein series. The local representation is $Ind_B^{GL_3}\chi\delta_B^{1/2}$. Here, $B$ is the Borel subgroup of $GL_3$ and $\chi$ is the character of the torus of $GL_3$ defined as

$$\begin{aligned}\chi(\mathrm{diag}(a,b,c)) &= |a|^{(4\nu-1)/3}|b|^{2(1-\nu)/3}|c|^{-2(\nu+1)/3}\\ &= |a|^{(4\nu-1)/3}|bc|^{(-4\nu+1)/6}|bc^{-1}|^{1/2}.\end{aligned}$$

From this, we deduce that the unramified component of the residue is a constituent of $Ind_Q^{GL_3}\chi'\delta_Q^{1/2}$ where $Q$ is the other standard maximal parabolic subgroup of $GL_3$ whose Levi part is $GL_1\times GL_2$. Also, for $(a,h)\in GL_1\times GL_2$ we have $\chi'((a,h)) = |a|^{(4\nu-1)/3}|deth|^{(-4\nu+1)/6}$. It is easy to see that this last induced representation is not generic. From this we deduce that $\mathcal{O}(E_\nu(g)) = (21)$. Hence we have proved that $\mathcal{O}_G^{\min}((2,1) = (21)$.

A somewhat more interesting example is

(2) Let $G = Sp_6$. Let $P$ denote the standard maximal parabolic subgroup of $G$ whose Levi part is $GL_2\times SL_2$. Let $\tau$ and $\sigma$ denote irreducible generic representations of $GL_2(\mathbf{A})$ and $SL_2(\mathbf{A})$, respectively. Let $E_{\tau,\sigma}(g,s)$ denote the

Eisenstein series corresponding to the induced representation $Ind_{P(\mathbf{A})}^{G(\mathbf{A})}(\tau\otimes\sigma)\delta_P^s$. In this example, we would like to compute the sets $\mathcal{O}_G^{\min}((2,2))$ and $\mathcal{O}_G^{\max}((2,2))$ and to determine the graph $\Gamma_G(E_{\tau,\sigma}(g,s))$ of this Eisenstein series. Since for $Re(s)$ large this Eisenstein series is generic, it follows that $\mathcal{O}_G^{\max}((2,2)) = (6)$. Since $\tau$ and $\sigma$ are generic, they are either cuspidal representations or Eisenstein series. Thus, if $\tau$ and $\sigma$ are unitary, then the poles of $E_{\tau,\sigma}(g,s)$ are determined by the poles of $L^S(\tau\times\sigma, 5(s-1/2))L^S(\tau,\wedge^2,5(2s-1))$. Here $S$ is a finite set of places, including the archimedean places, such that outside of $S$ all representations are unramified. It is well known that after fixing $\tau$ and $\sigma$, the representation $E_{\tau,\sigma}(g,s)$ cannot have a pole at $s_1=7/10$ and $s_2=6/10$ at the same time.

Assume that $\tau$ and $\sigma$ are both cuspidal representations and are such that $L^S(\tau\times\sigma,5(s-1/2))$ has a simple pole at $s_1$. This happens if both representations are the lift from some automorphic representation on $SO_2(\mathbf{A})$. Denote $E_{\tau,\sigma,s_1}(g) = Res_{s=s_1}E_{\tau,\sigma}(g,s)$. In particular, if $Ind_{B(GL_2)}^{GL_2}\mu_1\delta_{B(GL_2)}^{1/2}$ and $Ind_{B(SL_2)}^{SL_2}\mu_2\delta_{B(SL_2)}^{1/2}$ are the unramified induced representations corresponding to $\tau$ and $\sigma$, then $\mu_1(\mathrm{diag}(a,b)) = \chi(ab^{-1})$ and $\mu_2(\mathrm{diag}(a,a^{-1})) = \chi(a)$. Hence, if we write the unramified parameters of the local component of $E_{\tau,\sigma,s_1}$, we easily see that it is a constituent of $Ind_Q^G\chi'\delta_Q^{1/2}$. Here $Q$ is the maximal parabolic subgroup of $G$ whose Levi part is $GL_3$, and for $h\in GL_3$ we have $\chi'(h)=\chi(\det h)$. It is a matter of double coset calculation to show that this local induced representation does not support any local functional which corresponds to a global Fourier coefficient associated with the unipotent orbits that are greater than $(2^3)$. From this we deduce that $\mathcal{O}_G^{\min}((2,2))\le(2^3)$.

To verify Conjecture 5.6, we need to prove that $\mathcal{O}_G^{\min}((2,2))=(2^3)$. To do that, consider the integral

$$(16)\qquad \int_{(F\backslash A)^6} E_{\tau,\sigma}\left(\begin{pmatrix} 1 & & & x_1 & y_1 & y_2\\ & 1 & & y_3 & x_2 & y_1\\ & & 1 & y_4 & y_3 & x_1\\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1\end{pmatrix} g, s\right)\psi(x_1+x_2)dx_idy_j.$$

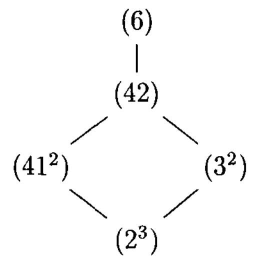
Here and below, we represent the group $G$ in matrices according to the form as defined in [G-J-R]. Since the stabilizer of $\psi$ inside $GL_3$ is $SO_3$, it follows that the Fourier coefficient (16) corresponds to the unipotent orbit $(2^3)$. We claim that there is a choice of data such that integral (16) is not zero. Assume not. Arguing in a similar way to [G-R-S4] Theorem 1 and Lemmas 1 and 2 on pages 889–898, we deduce that integral (16) is zero for each choice of data, if and only if the integral

(17)

$$\int_{(F\backslash A)^2}\int_{U(P)(F)\backslash U(P)(\mathbf{A})} E_{\tau,\sigma}\left(\begin{pmatrix} 1 & x & & & & \\ & 1 & & & & \\ & & 1 & y & & \\ & & & 1 & & \\ & & & & 1 & -x \\ & & & & & 1 \end{pmatrix} ug, s\right)\psi(x+y)dudxdy$$

is zero for each choice of data. Here, $U(P)$ is the unipotent radical of the parabolic subgroup $P$. Since $\tau$ and $\sigma$ are generic, it follows that integral (17) is nonzero for some choice of data.

So far, we have proved that $\mathcal{O}_G^{\max}((2,2) = (6)$ and $\mathcal{O}_G^{\min}((2,2) = (2^3)$. The relevant graph of all unipotent orbits for the group $Sp_6$ is given by




Besides $(41^2)$, all orbits are special. Next, we shall give an example so that $\mathcal{O}_G(E_{\tau,\sigma}(g,s)) = (42)$. To do that, let $\tau$ denote a self-dual cuspidal representation of $GL_2(\mathbf{A})$ and let $\sigma$ denote a cuspidal representation defined on $SL_2(\mathbf{A})$ such that $L^S(\tau\times\sigma, 1/2)$ is nonzero. From the above discussion about the poles of $E_{\tau,\sigma}(g,s)$, we deduce that it has a simple pole at the point $s_2 = 6/10$. We denote the residue by $E_{\tau,\sigma,s_2}(g)$. It follows from [G-J-R] Proposition 7.1 that this residue representation has a nonzero Fourier coefficient corresponding to the unipotent orbit $(41^2)$. Since this orbit is non-special, it follows that $E_{\tau,\sigma,s_2}(g)$ has a nonzero Fourier coefficient corresponding to the unipotent orbit (42). Since $E_{\tau,\sigma,s_2}(g)$ is not generic, it thus follows that $\mathcal{O}_G(E_{\tau,\sigma,s_2}) = (42)$.

At this point it is not clear whether there is a choice of data so that $\mathcal{O}_G(E_{\tau,\sigma,s_2})$ $= (3^2)$. There is some indication that it is possible, but we prefer not to speculate on this any further.

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